MATH 303 - MEASURES AND INTEGRATION
FINAL EXAM STUDY GUIDE

2. Measure Spaces

Main definitions. σ-algebras, Borel sets, measurable functions, measures

Main theorems. *measurability criteria for extended real-valued functions (Proposition 2.11)*, *basic properties of measures: monotonicity, countable subadditivity, continuity from below and above*

3. Integration

Main definitions. simple functions, integrable functions, definition of the integral, null sets, completion of measure spaces, convergence a.e., convergence in L^1

Main theorems. *approximation of measurable functions by simple functions*; basic properties of the integral: linearity, monotonicity, triangle inequality; _monotone convergence theorem_; _Fatou's lemma_; _dominated convergence theorem_; *_Borel–Cantelli lemma_*; *integration "ignores" null sets (Propositions 3.20, 3.22, and 3.23)*

4. Lebesgue–Stieltjes Measures

Main definitions. locally finite measure, distribution function, π-system, λ-system, semi-algebra, algebra, pre-measure, outer measure, measurable set (with respect to an outer measure), Lebesgue–Stieltjes measure, Lebesgue measure

Main theorems. _π-λ theorem_, _Carathéodory's theorem_, _Hahn–Kolmogorov extension theorem_, _existence and uniqueness of Lebesgue–Stieltjes measures_, existence of a Lebesgue non-measurable set, regularity properties of Lebesgue–Stieltjes measures

5. Radon Measures

Main definitions. locally compact, Hausdorff, support of a function, Radon measure, positive linear functional

Main theorems. _Urysohn's lemma_, partition of unity, _Riesz representation theorem_

6. Product Measures

Main definitions. σ-finite measure, s-finite measure, measurable rectangle, product σ-algebra, product measure, cross-sections of sets and functions, cross-sectional product measure, maximal product measure

Main theorems. existence of cross-sectional product of s-finite measures, existence of (maximal) product measure of arbitrary measures, *uniqueness of product of σ-finite measures*, _Fubini–Tonelli theorem_

7. L^p Spaces

Main definitions. topological vector space, norm, inner product, (topological) dual space, Banach space, L^p norm and L^p space, convex sets and functions

Main theorems. dual of a normed space is Banach, *Jensen's inequality*, *Minkowski's inequality*, *Hölder's inequality*, *Riesz–Fischer theorem*, Young's inequality

8. Littlewood's Principles

Main definitions. second countable topological space, σ-compact set, σ-finite set, regular Borel measure

Main theorems. inner regularity of Radon measures on σ-finite sets, regularity of locally finite Borel measures on second countable LCH spaces, Steinhaus's theorem, Riemann–Lebesgue lemma, *density of $C_c(X)$ in $L^p(\mu)$*, *Lusin's theorem* (for real or complex-valued functions), *Egorov's theorem* (for real or complex-valued functions)

9. Differentiation of Measures

Main definitions. signed and complex measures, null sets, positive sets, negative sets, absolutely continuous, mutually singular, Hahn decomposition, Jordan decomposition, positive variation, negative variation, total variation, Lebesgue decomposition, Radon–Nikodym derivative, integral against a signed/complex measure

Main theorems. *continuity properties for signed and complex measures*, *Hahn decomposition theorem*, *Jordan decomposition theorem*, *Lebesgue decomposition theorem*, *Radon–Nikodym theorem*

10. Mini-project topics

You should be familiar with one of the three topics below.

10.1. Countable product spaces.

Main definitions. sample space, event, random variable, expected value, independence (of events, σ-algebras, and random variables), product σ-algebras

Main theorems. *Kolmogorov extension theorem* (for probability measures on $\mathbb{R}^{\mathbb{N}}$)

10.2. Haar measure.

Main definitions. topological group, Haar measure, convolution of measures

Main theorems. *Haar's theorem*

10.3. Hausdorff dimension.

Main definitions. Hausdorff measures, Hausdorff dimension, similitude

Main theorems. *dimension of set invariant under a family of similitudes (Theorem 11)*

EXAM GUIDELINES AND FORMAT

- If an item is marked with asterisks (*), I expect that, given a precise mathematical statement, you can produce a proof of the result (using other theorems proved in the course as needed).
- For the main convergence theorems (monotone convergence, Fatou, and dominated convergence), I do not expect you to be able to prove any of them from scratch without using the others. However, I may ask you to use one of the convergence theorems to deduce another. For example, the proofs in the lecture notes of Fatou's lemma and the dominated convergence theorem (using monotone convergence and Fatou, respectively) are fair game, but I won't ask you to prove any convergence theorems straight from the definition of the integral.
- If an item is underlined and in italics, I expect that, given the name of the theorem, you can provide a precise formulation (but not necessarily its proof, unless the item is also marked with asterisks).
- I expect that you know the definitions of the objects in "main definitions".
- The final exam will consist of 3 sections:
  - 2 required problems testing your understanding of the main definitions and theorems
  - 3 problems on applying the main theorems (similar to exercise/homework questions). There will be a total of 4 problems given in this section, and you may choose which 3 to solve.
  - 1 problem connecting the mini-project topic with other results from the course. There will be 1 problem related to each topic, and you may choose which problem to solve.
- You may freely use any of the theorems listed above as well as any other results proved in lectures, homeworks, or exercises throughout the semester. When using a theorem, you should cite its name (e.g., "By Lusin's theorem, ...") or give a description of an unnamed theorem (e.g., "As we showed in class, Radon measures are inner regular on sets of finite measure, so ..."). The (hopefully obvious) exception to this rule is that if I ask you to prove a specific result, you cannot appeal to the proof of the same result in the lecture notes. (For example, if I ask you to deduce the dominated convergence theorem from Fatou's lemma, a "proof" that reads in full, "We proved the dominated convergence theorem using Fatou's lemma in lecture," will not receive any points.)

Practice Problems

The practice problems below are to give you a sense of the kinds of problems that will appear in the first two sections of the exam. For the third section of the exam, a good source of practice problems is to solve some additional exercises from the mini-project other than the ones you submitted for Homework 6.

You also have access to last year's exam on the Moodle page for the course. However, you should be aware that the format of the exam has changed slightly with the introduction of a new section related to the mini-projects, so last year's exam will only help directly with preparation for the first two sections of the exam.

Understanding main definitions and theorems. 2 required problems of this type will appear on the exam. Possible additional problems of this kind are to prove one of the results marked with asterisks.

Problem 1.

(a) State the monotone convergence theorem.
(b) State the dominated convergence theroem.
(c) Use the dominated convergence theorem to give a proof of the monotone convergence theorem. [**Hint:** If $\sup_{n\in\mathbb{N}} \int_X |f_n| \, d\mu < \infty$, then the set $\{x \in X : f_n(x) \neq 0 \text{ for some } n\}$ is a σ-finite set. Use this to reduce to the case that the measure space is finite.]

Problem 2. Let $(X, \mathcal{B})$, $(Y, \mathcal{C})$, and $(Z, \mathcal{D})$ be measurable spaces.

(a) Show that $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} = \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$ and that this σ-algebra is equal to the σ-algebra on $X \times Y \times Z$ generated by the family of "measurable boxes" $\{B \times C \times D : B \in \mathcal{B}, C \in \mathcal{C}, D \in \mathcal{D}\}$.
(b) Suppose $\mu : \mathcal{B} \to [0, \infty]$, $\nu : \mathcal{C} \to [0, \infty]$, and $\rho : \mathcal{D} \to [0, \infty]$ are σ-finite measures. Show that $(\mu \times \nu) \times \rho = \mu \times (\nu \times \rho)$ and that this measure is the unique measure on $\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}$ assigning a measure of $\mu(B)\nu(C)\rho(D)$ to each measurable box $B \times C \times D$.

Problem 3. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f \in L^1(\mu)$. Show that the following are equivalent:

(i) $f = 0$ a.e.;
(ii) $\int_X |f| \, d\mu = 0$;
(iii) $\int_E f \, d\mu = 0$ for every $E \in \mathcal{B}$.

Applying the main theorems. 4 problems of this type will appear on the exam, out of which you may choose which 3 to solve.

Problem 4. Let λ be the Lebesgue measure on $\mathbb{R}$. For $\varepsilon > 0$, let

$$A_\varepsilon = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}} \text{ for infinitely many } p, q \in \mathbb{Z} \text{ with } q \geq 1 \right\}.$$

Show that $\lambda(A_\varepsilon) = 0$ for every $\varepsilon > 0$.

[Note: A theorem of Dirichlet, which can be proved using the pigeonhole principle, says that every irrational number x can be approximated by rationals in such a way that $\left|x - \frac{p}{q}\right| < \frac{1}{q^2}$ for infinitely many $p, q \in \mathbb{Z}$ with $q \geq 1$. This problem shows that the exponent 2 is best possible for almost all numbers.]

Problem 5. Let $a \in (0,1)$ and $\varepsilon > 0$.

(a) Show that there exists $M \in \mathbb{N}$ with the following property: if $(X, \mathcal{B}, \mu)$ is a probability space and $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets such that $\inf_{n\in\mathbb{N}} \mu(A_n) = a$, then there exist $1 \le n < m \le M$ such that $\mu(A_n \cap A_m) > a^2 - \varepsilon$.

(b) Prove the following generalization for intersections of more sets. Let $k \in \mathbb{N}$. Show that there exists $M_k \in \mathbb{N}$ (depending also on a and ε) with the property: if $(X, \mathcal{B}, \mu)$ is a probability space and $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets such that $\inf_{n\in\mathbb{N}} \mu(A_n) = a$, then there exist $1 \le n_1 < n_2 < \cdots < n_k \le M_k$ such that

$$\mu\left(\bigcap_{j=1}^{k} A_{n_j}\right) > a^k - \varepsilon.$$

Problem 6. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ a measurable function. Prove Chebyshev's inequality: for every $c \in (0, \infty)$ and $p \in [1, \infty)$,

$$\mu(\{|f| > c\}) \le \left(\frac{\|f\|_p}{c}\right)^p$$

Problem 7. Let $E \subseteq \mathbb{R}$ be a Lebesgue-measurable set with $\lambda(E) > 0$. Fix a finite set $F \subseteq \mathbb{R}$. Show that E contains a homothetic copy of F, i.e. a set of the form $aF + b = \{af + b : f \in F\}$ with $a \neq 0$ and $b \in \mathbb{R}$.

[Note: There is a deep theorem in additive combinatorics, known as Szemerédi's theorem, that provides a strengthening to the conclusion; namely, one can bound the scaling factor $a > \delta$ for some δ depending on the set F and the size of the set E.]

Problem 8. For fixed $x \in \mathbb{R}$, let $L_x = \{(x, y) : y \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be the vertical line over x. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the second coordinate $\pi(x, y) = y$. Define

$$\tau = \{G \subseteq \mathbb{R}^2 : \pi(G \cap L_x) \text{ is open for every } x \in \mathbb{R}\}.$$

(a) Show that τ is a topology on $\mathbb{R}^2$ and $(\mathbb{R}^2, \tau)$ is a locally compact Hausdorff space.

(b) Prove that $K \subseteq \mathbb{R}^2$ is compact (with respect to τ) if and only if $\pi(K \cap L_x)$ is compact for every $x \in \mathbb{R}$ and $K \cap L_x = \emptyset$ for all but finitely many x.

(c) Define $\varphi : C_c(\mathbb{R}^2, \tau) \to \mathbb{C}$ by

$$\varphi(f) = \sum_{x\in\mathbb{R}} \int_{\mathbb{R}} f(x, y)\, dy,$$

where the integral with respect to y is the Riemann integral. Show that φ is a positive linear functional.

(d) Determine the measure μ representing φ.

Problem 9. For $x \in [0, 1)$, consider the binary expansion $x = \sum_{j=1}^{\infty} \frac{a_j(x)}{2^j}$ with $a_j(x) \in \{0, 1\}$. Let $f(x) = \min\{j \in \mathbb{N} : a_j(x) = 1\}$.

(a) Show that f is Borel-measurable.

(b) Compute the integral of f with respect to the Lebesgue measure on $[0, 1)$.

[Note: This problem has a probabilistic interpretation. Sampling $x \in [0, 1)$ randomly according to the Lebesgue measure, the sequence $a_1(x), a_2(x), a_3(x), \ldots$ is a sequence of independent fair coin flips (where we interpret 0 as tails and 1 as heads). With this interpretation, the value of $\int_{[0,1)} f\, d\lambda$ is the expected number of flips required until we see a result of heads.]

Problem 10. Let $(X, \mathcal{B}, \mu)$ be a measure space. Show that the following are equivalent:

(i) μ is s-finite;

(ii) there exists a finite measure ν such that $\mu \approx \nu$;

(iii) there exists a σ-finite measure ν such that $\mu \ll \nu$.

Problem 11. Let μ_1, μ_2 be finite positive measures on a measurable space $(X, \mathcal{B})$. Characterize the pairs (μ_1, μ_2) for which $(\mu_1 - \mu_2)^+ = \mu_1$ and $(\mu_1 - \mu_2)^- = \mu_2$.

Problem 12. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $A, B \in \mathcal{B}$. Define $\nu(E) = \mu(E \cap A) - \mu(E \cap B)$ for $E \in \mathcal{B}$.

(a) Show that ν is a signed measure.
(b) Determine the Hahn decomposition of ν.
(c) Show that $\nu \ll \mu$.
(d) Compute the Radon–Nikodym derivative $\frac{d\nu}{d\mu}$.

Problem 13. Let X be an LCH space, and let μ be a Radon measure on X. Show that there exists a closed set $C \subseteq X$ with the following two properties:

(i) $\mu(X \setminus C) = 0$, and
(ii) if $U \subseteq X$ is open and $U \cap C \neq \emptyset$, then $\mu(U) > 0$.

[Note: The set C is called the (topological) *support* of the measure μ.]

Problem 14. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Suppose ν_1, ν_2 are positive measures on $(X, \mathcal{B}, \mu)$ with $\nu_1(X) = \nu_2(X) = 1$ and $\nu_1, \nu_2 \ll \mu$. Show

$$\sup_{E \in \mathcal{B}} (\nu_1(E) - \nu_2(E)) = \frac{1}{2} \int_X \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| \, d\mu.$$

Problem 15. Let $A, B \subseteq [0,1)$ be Lebesgue-measurable sets. For each $t \in [0,1)$, let $B_t = \{b + t \mod 1 : b \in B\}$. Show that there exists $t \in [0,1)$ such that $\lambda(A \cap B_t) \geq \lambda(A)\lambda(B)$.

Problem 16. Let X be a compact metric space, and let $T : X \to X$ be a continuous function. We say that a probability measure $\mu : \text{Borel}(X) \to [0, \infty]$ is T-*invariant* if $\mu(T^{-1}E) = \mu(E)$ for every $E \in \text{Borel}(X)$. Denote by $\mathcal{M}(X,T)$ the space of T-invariant Borel probability measures on X.

(a) Show that $\mathcal{M}(X,T)$ is a convex set.

Given a convex set C, a point $x \in C$ is an *extreme point* if the only solution to $x = ty + (1-t)z$ for $t \in (0,1)$ and $y, z \in C$ is $y = z = x$.

(b) Let $\mu \in \mathcal{M}(X,T)$. Show that the following are equivalent:
    (i) μ is an extreme point of $\mathcal{M}(X,T)$.
    (ii) if $E \in \mathcal{B}$ and $\mu\left(E \triangle T^{-1}E\right) = 0$, then $\mu(E) \in \{0,1\}$.

    [**Hint for (ii) $\implies$ (i):** Suppose μ satisfies (ii), and write $\mu = t\nu_1 + (1-t)\nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}(X,T)$ and $t \in (0,1)$. Let $f = \frac{d\nu_1}{d\mu}$ and consider $E = \{f < 1\}$. Show that $\int_{E \setminus T^{-1}E} f \, d\mu = \int_{T^{-1}E \setminus E} f \, d\mu$ and deduce that $f = 1$ a.e.]

A measure satisfying (ii) is called *ergodic*. Let $\mathcal{E}(X,T)$ denote the set of ergodic T-invariant Borel probability measures on X.

(c) Suppose $\mu, \nu \in \mathcal{E}(X,T)$ and $\mu \neq \nu$. Show that $\mu \perp \nu$.